EPFL

Probability and Statistics for IC
2023, Fall semester

Do not turn this page until instructed

Probability and Statistics: Midterm

November 7, 2023

Duration: The exam will start at 8:20 and end at 10:00 (unless special arrangements).

Family name:

First name:

SCIPER number:

PROTOCOL:

- If caught cheating, you will get a 0 and we will report to the section.
- No personal documents, cheat sheets or calculators are allowed during the exam.
- Justify all your answers! Unjustified answers will not get full score even if correct. However, partial reasoning might get partial points.
- Try to simplify numerical expressions but no need to give exact decimal expressions (e.g., you can leave factorial expressions).

Exercise 1 (7.5 points)**.** At the end of a video game, the player must kill a monster sampled randomly as follows:

- 1 time out of 10, it is a *dragon*,
- 3 times out of 10, it is a *troll*,
- the rest of the time, it is a *giant*.

When the monster dies, the player gets a chance to get a *ruby*:

- the dragon always gives a ruby,
- the troll gives a ruby 1 time out of 2,
- the giant gives a ruby 1 time out of 5.

We assume that the game is very easy; thus the players always succeed at killing the monster. Different rounds of the game are independent.
Alice plays the game. We denote D (respectively T, G) the event "the monster is a dragon" (respectively a troll, a giant). We denote R the event "Alice wins a ruby", and $p = \mathbb{P}(R)$.

1. (2 pt) Compute p.

2. (1 pt) Alice won a ruby! What is the probability that the monster was a troll?

3. (2.5 pt) Bob decides to play 6 games. We denote S the number of rubies that he gets. (We recommend you to express your answers in p as much as possible.)

  (a) (1 pt) What is the distribution of S?
  (b) (0.5 pt) What is the expectation of S?
  (c) (0.5 pt) What is the probability that Bob wins exactly 3 rubies?
  (d) (0.5 pt) What is the probability that Bob wins at least 1 ruby?

4. (2 pt) Charlie decides to play until he wins one ruby. We denote X the number of games Charlie plays.

  (a) (1 pt) What is the distribution of X?
  (b) (0.5 pt) What is the expectation of X?
  (c) (0.5 pt) What is the probability that Charlies does exactly 3 games?

Exercise 2 (4 points)**.** Let (X, Y) be a joint random variable whose probability mass function is given by the following table:

X \ Y	-1	1
0	0.2	0.4
1	0.3	0.1

1. (1 pt) What are the marginal distributions of X and Y?

2. (1 pt) Are X and Y independent? (As always, justify your answer.)

2. (2 pt) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[X+Y]$ and $\mathbb{E}[XY]$.

Exercise 3 (6 points)**.** We consider 5 urns: 3 of type A and 2 of type B. Each urn of type A contains 1 red ball and 3 white balls; each urn of type B contains 2 red balls and 2 white balls. We sample an urn uniformly at random; then, in the chosen urn, we sample two balls uniformly at random, *with replacement* (i.e., we replace the first sampled ball back in the urn before sampling the second ball).
Let U_A (resp. U_B) denote the event "the chosen urn is of type A (resp. of type B)". Let X denote number of red balls among the sampled balls.

1. (1.5 pt) Compute $\mathbb{P}(U_A)$, $\mathbb{P}(U_B)$ and $\mathbb{P}(X = k|U_B)$ for $k = 0, 1, 2$.

2. (1.5 pt) What is the probability mass function of X?

3. (1 pt) Compute $\mathbb{E}[X]$.

4. (1 pt) Knowing that we have sampled a red ball and a white ball, what is the probability that we have sampled an urn of type A?

5. (1 pt) Are the events $\{X = 1\}$ and U_A independent? As always, justify your answer.

Exercise 4 (Multiple Choice Question; +3 if the correct and −1 if a wrong choice is selected.)**.** Let $X \sim \text{Uniform}(1, 3)$ and Y such that $Y|X \sim \exp(X)$, i.e., $f_{Y|X}(y|x) = xe^{-yx}$ for $y \in (0, \infty), x \in (1, 3)$. What is the value of $\mathbb{E}[X^2Y]$?

(a) 1

(b) 2

(c) 4

(d) 9

(The above (and only the above) question is a multiple choice question, you do not need to deliver a solution.)

Exercise 5 (5.5 points)**.** Let $F : \mathbb{R} \to [0, 1]$ be the function defined by

$$F(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ x^2 & \text{if } 0 \le x \le 1\,, \\ 1 & \text{if } 1 < x\,. \end{cases}$$

1. (1.5 pt) Justify that F is a cumulative distribution function.

2. (0.5 pt) Justify that the distribution associated to F is continuous.

3. (1 pt) Compute the density f associated to this distribution.

4. (2.5 pt) Let X be a random variable with cumulative distribution function F.

- **(a)** (0.5 pt) Compute $\mathbb{P}(X > 1/2)$.
- **(b)** (1 pt) Compute $\mathbb{E}[X]$.
- **(c)** (1 pt) Compute the variance of X.

Exercise 6 (2.5 points)**.** Let $(X, Y) \in \mathbb{R}^2$ be a random point, sampled uniformly in the unit disk. Said differently, (X, Y) has density

$$f_{(X,Y)}(x, y) = \frac{1}{\pi} I(x^2 + y^2 \le 1) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \le 1\,, \\ 0 & \text{if } x^2 + y^2 > 1\,. \end{cases}$$

1. (1.5 pt) What is the density of X?

2. (1 pt) Are X and Y independent? (As always, justify your answer.)

Exercise 7 (6 points)**.** *This exercise is significantly harder than the previous ones. Try it once you have finished the rest of the midterm.*
We remind that:

- *an exponential random variable with rate* λ *has density* $f(x) = \lambda e^{-\lambda x} I(x > 0)$,
- *a gamma random variable has with shape parameter* α *and rate* λ *has density*

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I(x > 0)\,,$$

- *if* n *is a non-negative integer,* $\Gamma(n+1) = n!$, *and*
- *Poisson random variable with rate* λ *has probability mass function* $f(n) = e^{-\lambda}\lambda^n/n!$, $n = 0, 1, \ldots$.

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent random variables of exponential distribution with rate parameter 1. We denote $T_0 = 0$ and for all $n \geq 1$, $T_n = X_1 + \cdots + X_n$. For all $t \geq 0$, we denote $N_t = \max\{n \geq 0 \,|\, T_n \leq t\}$.

1. (2 pt)

(a) (1 pt) Compute the cumulative distribution function of T_2. Deduce that T_2 is a gamma random variable with shape parameter 2 and rate 1. (Like everywhere else, a proof is required.)

(b) (1 pt) Let $n \geq 1$. Using the same method, compute the distribution of T_n.

2. (2 pt) Let $t > 0$. Compute the distribution of N_t.

3. (2 pt) Let $n \geq 1$.

(a) (1 pt) Compute the joint distribution of $(T_1, \ldots, T_n)$.

(b) (1 pt) Compute the conditional distribution of $(T_1, \ldots, T_n)$ given that $N_t = n$.